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States on Projection Logics of von Neumann Algebras

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We summarize recent results concerning states on projection lattices of von Neumann algebras. In particular, we present an analysis of the Jauch–Piron property in the von Neumann algebra setting.

1. INTRODUCTION

In this paper we deal with various types of states on von Neumann algebras which are relevant to noncommutative measure theory and axiomatics of quantum physics. We show that certain properties of states, familiar within classical theory, have rather unexpected characterization in the noncommutative framework. As a by-product our study may provide an *a posteriori* physical justification of some axioms of quantum theory which seems to have been adopted basically for the sake of mathematical convenience. We have restricted ourselves mainly to Jauch–Piron states.

Let us first recall a few notions and fix notations. Throughout the paper, let $\mathscr{A}$ be a von Neumann algebra without type I_2 direct summand. An algebra $\mathscr{A}$ is said to be locally σ-finite if it is a direct sum of σ-finite algebras. The symbol $P(\mathscr{A})$ will denote the projection logic of $\mathscr{A}$. Let ϱ be a finitely additive state on $P(\mathscr{A})$. We say that ϱ is Jauch–Piron if $\varrho(e \vee f) = 0$ whenever $e, f \in P(\mathscr{A})$ with $\varrho(e) = \varrho(f) = 0$. Jauch–Piron states have been studied extensively (e.g., Beltrametti and Cassinelli, 1981; Bunce *et al.*, 1985; Jauch, 1968; Pták and Pulmannová, 1991; Rüttimann, 1977). A state ϱ is said to be regular if $\varrho(\sum e_n) = 0$ whenever (e_n) is an orthogonal sequence in $P(\mathscr{A})$ with $\varrho(e_n) = 0$ for all n. It turns out (Bunce and Hamhalter, 1993) that ϱ is regular if and only if ϱ fulfills the σ-Jauch–Piron condition: $\varrho(\bigvee e_n) = 0$ whenever (e_n) is any sequence in $P(\mathscr{A})$ with

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$\varrho(e_n) = 0$ for all n. Further, ϱ is said to be nonsingular if there is no orthogonal family (e_α) in $P(\mathscr{A})$ such that $\sum e_\alpha = 1$ and $\varrho(e_\alpha) = 0$ for all α. A state ϱ is called subadditive if $\varrho(e \vee f) \leq \varrho(e) + \varrho(f)$ whenever $e, f \in P(\mathscr{A})$. Finally, ϱ is said to be tracial if it is unitarily invariant. (In other words, ϱ is tracial if it is a restriction of a finite trace of $\mathscr{A}$.) Let us recall that every state of $P(\mathscr{A})$ can be identified with a linear state of $\mathscr{A}$ by the deep Gleason–Christensen–Yeadon theorem (Gleason, 1957; Christensen, 1982; Yeadon, 1984).

2. PURE JAUCH–PIRON STATES

In this section we analyze pure Jauch–Piron states. A striking result in this direction is a recent result of Amann (1989) saying that pure Jauch–Piron states on σ-finite algebras have to be normal. Relaxing the σ-finiteness, L. J. Bunce and the author succeeded in generalizing this result for all von Neumann algebras. The result provides the following characterization of pure Jauch–Piron states.

Theorem 2.1 (Bunce and Hamhalter, 1993). Let $\mathscr{A}$ have no commutative direct summand. Then a pure state of $P(\mathscr{A})$ is Jauch–Piron if and only if it is σ-additive.

As a consequence of this theorem one can give the following criterion of complete additivity: A pure state on $P(\mathscr{A})$ is completely additive if and only if it is a Jauch–Piron state not vanishing on all σ-finite projections.

Restricting ourselves to Hilbert spaces of appropriate dimensions, we can present a lucid description of pure Jauch–Piron states as follows. (Let us recall that a cardinal is real-nonmeasurable if its power set does not admit any nonzero probability measure vanishing on one-point sets.)

Theorem 2.2 (Hamhalter, 1993). Let $\mathscr{A}$ be a von Neumann algebra acting on a Hilbert space H with real-nonmeasurable dimension. Let ϱ be pure. Then there is a central projection, $z \in P(\mathscr{A})$, such that $\varrho(z) = 1$ and such that one of the following conditions is fulfilled:

(i) $zP(\mathscr{A})$ is a Boolean algebra,

(ii) $zP(\mathscr{A})$ is isomorphic to the algebra of all bounded operators acting on $z(H)$ and

$$\varrho(e) = \|e\xi\|^2 \quad [e \in zP(\mathscr{A})] \qquad \text{for some unit vector} \quad \xi \in z(H)$$

In axiomatics of quantum theory it is often contended that a "state" of a physical system should be represented by a Jauch–Piron state defined on $P(\mathscr{A})$ (e.g., Beltrametti and Cassinelli, 1981; Jauch, 1968; Jauch and Piron, 1963). In this context the previous theorem has the following physical interpretation: Every individual pure state of a quantum system is either given by a state of the Boolean part of the system (as in the classical Kolmogorov probability model) or it is represented by a vector state on a Hilbert space logic (as in the traditional Hilbert space formalism of quantum mechanics). This result also advocates the central position of the Hilbert-space logic in the algebraic quantum mechanical approach.

3. TWO-VALUED STATES

Let ϱ be a two-valued state on $P(\mathscr{A})$. (A state is said to be two-valued if it attains only the values 0 and 1.) It turns out that in this case ϱ has to be Jauch–Piron. Employing the results about the pure Jauch–Piron states, we can then state the following characterization of two-valued states on von Neumann algebras.

Theorem 3.1 (Hamhalter, 1993). Let ϱ be a two-valued state of $\mathscr{A}$. Then there is a central projection z in $\mathscr{A}$ such that $\varrho(z) = 1$ and $zP(\mathscr{A})$ is a Boolean algebra.

In other words, the two-valued states (the "hidden variables" in some quantum physics terminology) live on the classical part of the system only.

4. GENERAL JAUCH–PIRON STATES

In this section we present a characterization of Jauch–Piron states in terms of regularity. The first result in this context was given by Amann (1989), who proved that every Jauch–Piron state on a σ-finite factor has to be nonsingular. We summarize here results of Bunce and Hamhalter (1993), where a systematic investigation of the regularity of Jauch–Piron states has been conducted. (Amann's result can be viewed then as a consequence of the following fairly general theorems.)

Theorem 4.1 (Bunce and Hamhalter, 1993). Let $\mathscr{A}$ satisfy one of the following conditions:

(i) $\mathscr{A}$ is a factor.

(ii) $\mathscr{A}$ has no locally σ-finite direct summand.

(iii) $\mathscr{A}$ is a σ-finite type III.

Then a state ϱ of $P(\mathscr{A})$ is Jauch–Piron if and only if it is regular.

Moreover, if $\mathscr{A}$ is σ-finite, then ϱ is Jauch–Piron if and only if ϱ has a support.

As can be demonstrated by examples (Bunce and Hamhalter, 1993), these results cannot be extended to all von Neumann algebras. Nevertheless, there is a close relation between the Jauch–Piron property and nonsingularity on centers of hereditary subalgebras. For instance, we have the following result:

Theorem 4.2 (Bunce and Hamhalter, 1993). Let ϱ be a Jauch–Piron state of $P(\mathscr{A})$. Then ϱ is regular if it is nonsingular on centers of all hereditary sublogics $P_e(\mathscr{A})$ $[=\{f\in P(\mathscr{A}); f\leq e\}]$, where e is a projection with locally σ-finite central cover.

More specifically, if $\mathscr{A}$ is σ-finite, then ϱ is regular if and only if it is nonsingular on centers of all sublogics $P_e(\mathscr{A})$, where e is a finite projection.

Let us remark in the conclusion of this paper that the subadditivity of states can be viewed as a strong form of the Jauch–Piron property. It turns out, perhaps surprisingly, that the only subadditive states are the tracial states. (This result is valid also for JBW-algebras.) This provides a new type of characterization of semifinite normal traces in terms of measure theory. (These results have been obtained recently by L. J. Bunce and the author and will be published in a subsequent paper.)

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